Constructing new complementarity functions from existing ones

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Abstract. In this short report, we survey concepts and properties of various types of complementarity functions, including NCP-functions, SOCCP-fucntions, and SCCP-functions. In addition, we provide an idea for constructing new complementarity functions from existing ones.

Keywords: NCP, SOCCP, SCCP, complementarity function.

1 Introduction

The complementarity problem arises from the KKT conditions of an optimization problem. Formally, it seeks to find an element x such that

$$x \succeq_{\mathcal{K}} 0, \quad F(x) \succeq_{\mathcal{K}} 0, \quad \langle x, F(x) \rangle = 0, \tag{1}$$

where $\mathcal{K}$ is usually a symmetric cone [14], $\succeq_{\mathcal{K}}$ is the partial order associated with $\mathcal{K}$, and $\langle \cdot, \cdot \rangle$ is an appropriate inner product. When $\mathcal{K}$ is the nonnegative orthant, the above problem (1) reduces to the well known nonlinear complementarity problem (NCP for short) which consists in finding a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \ldots, F_n)^T$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. NCPs have wide applicability in the fields of economics, engineering, and operations research, see [13, 15, 21] and references therein. When $\mathcal{K}$ represents a positive semidefinite cone $\mathcal{S}^n_+$, the complementarity problem (1) reduces to a semidefinite complementarity problem (SDCP for short). When $\mathcal{K}$ is the second-order cone (SOC) whose definition will be introduced later, the complementarity problem (1) is the second-order cone complementarity problem (SOCCP for short). All the above special cases can be unified as symmetric cone complementarity problem (SCCP) under Euclidean Jordan algebra.

Besides the symmetric cone complementarity problem which is endowed in a finite dimensional space, we further consider the generalized complementarity problem (GCP for short) in infinite dimensional space. More specifically, let $(X, \|\cdot\|)$ denote a real Banach space, X^* represent its dual space, we consider a cone K which is solid (i.e., $\text{int}K \neq \emptyset$) closed convex in X. Note that its dual cone K^+ is defined as

$$K^+ = \{x^* \in X^* : \langle x, x^* \rangle \geq 0, \ \forall x \in K\}.$$

In contrast to the aforementioned symmetric cone, K is not self-dual in general. Let $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ be the canonical bilinear pairing and $F : X \to X^*$. The generalized complementarity problem (GCP) is to find an element $x \in X$ such that

$$x \in K, \quad F(x) \in K^+, \quad \langle x, F(x) \rangle = 0. \tag{2}$$

The GCP was originally proposed by Karmardian in 1971, see [27]. For more details regarding GCP including solution methods, properties, and applications, please refer to the textbook [26].

To deal with various complementarity problems, the so-called complementarity functions (C-functions) play crucial roles in designing solution methods, see [3, 4, 7, 9, 16, 18, 25, 32] and the reference therein. In the setting of NCP, the complementarity function is abbreviated as NCP-function, which is denoted by $\phi : \mathbb{R}^2 \to \mathbb{R}$ and defined as

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a, b \geq 0, \ ab = 0.$$

Many NCP-functions and merit functions have been explored and proposed in many literature, see [20] for a survey. Among them, the Fischer-Burmeister (FB) function and the Natural-Residual (NR) function are two famous and effective NCP-functions. The FB function $\phi_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\text{FB}}(a, b) = \sqrt{a^2 + b^2} - (a + b), \ \forall (a, b) \in \mathbb{R}^2$$

and the NR function $\phi_{\text{NR}} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\text{NR}}(a, b) = a - (a - b)_+ = \min\{a, b\}, \ \forall (a, b) \in \mathbb{R}^2.$$

During the past four decades, approximately thirty NCP-functions have been proposed, see [20] for a survey. Some of them have been extending to be a complementarity functions for symmetric cone complementary problem, including SDCP, SOCCP. Among the existing NCP-functions, it is observed that none of them is both convex and differentiable. Miri and Effati [31] show that convexity and differentiability cannot hold simultaneously for an NCP-function. Huang et. al [22] further generalized this result for general complementarity functions associated with the GCP. In this article, we survey some newly discovered complementarity functions and provide an idea for constructing new complementarity functions from existing ones.

2 Preliminaries

In this section, we review the basic concepts and properties concerning Jordan algebras and symmetric cones from the book [14] which are needed in the subsequent analysis. Especially, we recall some background materials regarding second-order cone as well.

A *Euclidean Jordan algebra* is a finite dimensional inner product space $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ ($\mathbb{V}$ for short) over the field of real numbers $\mathbb{R}$ equipped with a bilinear map $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$;

(iii) $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ for all $x, y, z \in \mathbb{V}$,

where $x^2 := x \circ x$, and $x \circ y$ is called the *Jordan product* of x and y. Moreover, if there is an (unique) element $e \in \mathbb{V}$ such that $x \circ e = x$ for all $x \in \mathbb{V}$, the element e is called the *identity element* in $\mathbb{V}$. Note that a Jordan algebra does not necessarily have an identity element. Throughout this paper, we assume that $\mathbb{V}$ is a Euclidean Jordan algebra with an identity element e.

In a Euclidean Jordan algebra $\mathbb{V}$, the set of squares $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ is called a *symmetric cone* [14, Theorem III.2.1], which means $\mathcal{K}$ is a self-dual closed convex cone and, for any two elements $x, y \in \mathrm{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(x) = y$ and $\Gamma(\mathcal{K}) = \mathcal{K}$. An element $c \in \mathbb{V}$ is called an *idempotent* if $c^2 = c$, and it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. The idempotents c, d are said to be *orthogonal* if $c \circ d = 0$. In addition, a finite set $\{e^{(1)}, e^{(2)}, \cdots, e^{(r)}\}$ of primitive idempotents in $\mathbb{V}$ is said to be a *Jordan frame* if

$$e^{(i)} \circ e^{(j)} = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \sum_{i=1}^{r} e^{(i)} = e.$$

With the above, there has the spectral decomposition of an element x in $\mathbb{V}$.

Theorem 2.1. (Spectral Decomposition Theorem) [14, Theorem III.1.2] *Let $\mathbb{V}$ be a Euclidean Jordan algebra. Then there is a number r such that, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e^{(1)}, \cdots, e^{(r)}\}$ and real numbers $\lambda_1(x), \cdots, \lambda_r(x)$ with*

$$x = \lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}.$$

Here, the numbers $\lambda_i(x)$ $(i = 1, \cdots, r)$ are called the spectral values of x, the expression $\lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}$ is called the spectral decomposition of x. Moreover, $\mathrm{tr}(x) := \sum_{i=1}^{r} \lambda_i(x)$ is called the trace of x, $\det(x) := \lambda_1(x)\lambda_2(x)\cdots\lambda_r(x)$ is called the determinant of x, and r is called the rank of $\mathbb{V}$.

The second-order cone (SOC for short) in $\mathbb{R}^n$, also called the Lorentz cone, is defined by

$$\mathcal{K}^n = \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x_2\| \leq x_1 \right\}.$$

While $n = 1$, $\mathcal{K}^n$ denotes the set of nonnegative real number $\mathbb{R}_+$. For any $x, y \in \mathbb{R}^n$, we write $x \succeq_{\mathcal{K}^n} y$ if $x - y \in \mathcal{K}^n$, and $x \succ_{\mathcal{K}^n} y$ if $x - y \in \text{int}(\mathcal{K}^n)$. The relation $\succeq_{\mathcal{K}^n}$ is a partial ordering but not a linear ordering in $\mathcal{K}^n$. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define their *Jordan product* as

$$x \circ y = (x^T y \;,\; y_1 x_2 + x_1 y_2).$$

Then, $(\mathbb{R}^n, \circ, \langle \cdot, \cdot \rangle)$ forms a Euclidean Jordan algebra with identity $e = (1, 0, \ldots, 0)^T$. Notice that this Jordan product is *not associative*. However, it is power associative, i.e., $x \circ (x \circ x) = (x \circ x) \circ x$ for all $x \in \mathbb{R}^n$. Without loss of ambiguity, we may write x^m for the product of m copies of x and $x^{m+n} = x^m \circ x^n$ for all positive integers m and n. Here, we set $x^0 = e$.

For any $x \in \mathcal{K}^n$, it is known that there exists a unique vector in $\mathcal{K}^n$ denoted by $x^{1/2}$ such that $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$. Indeed,

$$x^{1/2} = \left(s, \frac{x_2}{2s} \right), \quad \text{where } s = \sqrt{\frac{1}{2}\left(x_1 + \sqrt{x_1^2 - \|x_2\|^2} \right)}.$$

In the above formula, the term x_2/s is defined to be the zero vector if $s = 0$, i.e., $x = 0$. Since $x^2 \in \mathcal{K}^n$ for any $x \in \mathbb{R}^n$, there exists a unique vector $(x^2)^{1/2} \in \mathcal{K}^n$, denoted by $|x|$. It is easy to verify that $|x| \succeq_{\mathcal{K}^n} 0$ and $x^2 = |x|^2$ for any $x \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$, we define $[x]_+$ to be the nearest point projection of x onto $\mathcal{K}^n$, which is the same definition as in $\mathbb{R}^n_+$. In other words, $[x]_+$ is the optimal solution of the parametric SOCP: $[x]_+ = \arg\min\{\|x - y\| \,|\, y \in \mathcal{K}^n\}$. In addition, it can be verified that $[x]_+ = (x + |x|)/2$; see [14, 19].

Optimization problems involved second-order cones have been appeared in real world applications. For dealing with second-order cone programs (SOCP) and second-order cone complementarity problems (SOCCP), there needs *spectral decomposition* associated with SOC [5]. More specifically, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the vector x can be decomposed as

$$x = \lambda_1 u_x^{(1)} + \lambda_2 u_x^{(2)}, \tag{3}$$

where λ_1, λ_2 and $u_x^{(1)}, u_x^{(2)}$ are the spectral values and the associated spectral vectors of x, respectively, given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|, \tag{4}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}(1, (-1)^i \frac{x_2}{\|x_2\|}) & \text{if} \quad x_2 \neq 0, \\ \frac{1}{2}(1, (-1)^i w) & \text{if} \quad x_2 = 0, \end{cases} \tag{5}$$

for $i = 1, 2$ with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition is unique. Accordingly, the determinant, the trace, and the Euclidean norm of x can all be represented in terms of λ_1 and λ_2:

$$\det(x) = \lambda_1\lambda_2, \quad \mathrm{tr}(x) = \lambda_1 + \lambda_2, \quad \|x\|^2 = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right).$$

From the simple calculation, we especially point out that $\mathrm{tr}(x) = 2x_1$, which we frequently use in the following paragraphs.

For any real valued function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}^n$ $(n \geq 1)$ was considered in [2, 6]:

$$f^{\mathrm{soc}}(x) = f(\lambda_1)u_x^{(1)} + f(\lambda_2)u_x^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{6}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition (6) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. The cases of $f^{\mathrm{soc}}(x) = x^{1/2}$, x^2, $\exp(x)$ are discussed in [14]. For subsequent analysis, we will frequently use the vector-valued functions corresponding to t^p $(t > 0, p > 0)$ and $|t|^p$ $(t \in \mathbb{R}, p > 0)$, respectively. In particular, they can be expressed as

$$\begin{aligned} x^p &= \lambda_1^p u_x^{(1)} + \lambda_2^p u_x^{(2)}, \quad \forall x \in \mathcal{K}^n, \\ |x|^p &= |\lambda_1|^p u_x^{(1)} + |\lambda_2|^p u_x^{(2)}, \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

The spectral decomposition along with the Jordan algebra associated with SOC entail some basic properties as listed in the following text. We omit the proofs since they can be found in [2, 14, 19].

3 New complementarity functions

Recently, there have many new NCP-functions as below which are newly discovered. In fact, they are constructed from existing ones. Inspired by this, we will provide another new idea for constructing complementarity functions.

(i) $\phi^p_{\mathrm{NR}}(a, b) = a^p - (a - b)^p_+$, $p > 1$ is odd integer (Chen, Ko, and Wu [8]).

(ii) $\phi^p_{\mathrm{S-NR}}(a, b) = \begin{cases} a^p - (a-b)^p & \text{if} \quad a > b, \\ a^p = b^p & \text{if} \quad a = b, \\ b^p - (b-a)^p & \text{if} \quad a < b, \end{cases}$ $p > 1$ is odd integer (Chang, Chen, and Yang [1]).

(iii) $\psi^p_{\mathrm{S-NR}}(a, b) = \begin{cases} a^p b^p - (a-b)^p b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ a^p b^p - (b-a)^p a^p & \text{if} \quad a < b, \end{cases}$ $p > 1$ is odd integer (Chang, Chen, and Yang [1]).

(iv) $\phi^p_{\text{D-FB}}(a,b) = \left(\sqrt{a^2+b^2}\right)^p - (a+b)^p$, $p>1$ is odd integer (Ma, Chen, Huang, and Ko [30]).

(v) $\varphi^p_{\text{NR}}(a,b) = \left(\dfrac{a+b}{2}\right)^p - \left(\dfrac{|a-b|}{2}\right)^p = \dfrac{1}{2^p}\left[(a+b)^p - |a-b|^p\right]$, $p>1$ is odd integer (Su [34]).

For more properties of the aforementioned NCP-functions, please refer to [24, 34].

Some of the above NCP-functions have been extended to symmetric cone and second-order cone settings. Here we define a monotone transformation that makes new complementarity functions (C-functions) from existing ones. We first formulate the general construction principle, which is called θ-extension.

Lemma 3.1. *Let $\mathcal{K}^n$ be a second-order cone and $x,y\in\mathbb{R}^n$. Suppose $\theta:\mathbb{R}\to\mathbb{R}$ a strictly monotone increasing and continuous function. Then, $x=y$ if and only if $\theta^{\rm soc}(x)=\theta^{\rm soc}(y)$.*

Proof. For $x=y$, it is clear that $\theta^{\rm soc}(x)=\theta^{\rm soc}(y)$ by spectral decomposition and the definition of the vector-valued function associated with symmetric cone.

For the other direction, suppose that $\theta^{\rm soc}(x)=\theta^{\rm soc}(y)$. Applying the spectral decomposition (3)-(5) give

$$\begin{aligned} x &= \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)},\\ y &= \lambda_1(y)u_y^{(1)} + \lambda_2(y)u_y^{(2)}. \end{aligned}$$

Then, we have $\theta^{\rm soc}(x)=\theta(\lambda_1(x))u_x^{(1)}+\theta(\lambda_2(x))u_x^{(2)}$ and $\theta^{\rm soc}(x)=\theta(\lambda_1(x))u_x^{(1)}+\theta(\lambda_2(x))u_x^{(2)}$. Since $\theta^{\rm soc}(x)=\theta^{\rm soc}(y)$ and the spectral values are unique, we have $\theta(\lambda_i(x))=\theta(\lambda_i(y))$ for $i=1,2$. By the strictly monotonicity and continuity of θ, we can conclude that $\lambda_i(x)=\lambda_i(y)$ for $i=1,2$. Besides, both $\{u_x^{(1)},u_x^{(2)}\}$ and $\{u_y^{(1)},u_y^{(2)}\}$ are Jordan frames, we further have $u_x^{(1)}+u_x^{(2)}=u_y^{(1)}+u_y^{(2)}=e$, where e is the identity. From $\theta^{\rm soc}(x)=\theta^{\rm soc}(y)$ and $u_x^{(1)}+u_x^{(2)}=u_y^{(1)}+u_y^{(2)}$, we may obtain

$$\left(\theta(\lambda_1(x))-\theta(\lambda_2(x))\right)\left(u_x^{(1)}-u_y^{(1)}\right)=0.$$

If $\theta(\lambda_1(x))=\theta(\lambda_2(x))$, we get $\lambda_1(x)-\lambda_2(x)$ and $\lambda_1(y)-\lambda_2(y)$, which imply that $x=\lambda_1 e=y$. Otherwise, we must have $u_x^{(1)}=u_y^{(1)}$, and hence $u_x^{(2)}=u_y^{(2)}$. Therefore, we complete the proof. $\square$

Proposition 3.1. *Assume that $\phi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous and $\phi(x,y)=f_1(x,y)-f_2(x,y)$. Let $\theta:\mathbb{R}\to\mathbb{R}$ be a strictly monotone increasing and continuous function. Then, ϕ is an C-function associated with the second-order cone if and only if $\phi_\theta(x,y)=\theta^{\rm soc}(f_1(x,y))-\theta^{\rm soc}(f_2(x,y))$ is an C-function associated with second-order cone.*

Proof. By Lemma 3.1, we have

$$\begin{aligned} & \phi(x,y) = 0 \\ \Longleftrightarrow \quad & f_1(x,y) = f_2(x,y) \\ \Longleftrightarrow \quad & \theta^{\mathrm{soc}}(f_1(x,y)) = \theta^{\mathrm{soc}}(f_2(x,y)) \\ \Longleftrightarrow \quad & \phi_\theta(x,y) = 0. \end{aligned}$$

Hence, we obtain the conclusion. □

Note that from Proposition 3.1, some new complementarity functions associated with $\mathcal{K}^n$ appeared in [30, Propsition 4.1-4.2] can be deduced by setting $\theta(t) = t^p$ with positive odd integer p. Applying Proposition 3.1 to φ^p_{NR}, we achieve a new complementarity functions associated with $\mathcal{K}^n$ via defining $\varphi^p_{\mathrm{NR}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\varphi^p_{\mathrm{NR}}(x,y) = \frac{(x+y)^p - |x-y|^p}{2^p} \tag{7}$$

where $p > 1$ is a positive odd integer, $x, y \in \mathbb{R}^n$. It is clear that this C-function is symmetric in x and y, that is, $\varphi^p_{\mathrm{NR}}(x,y) = \varphi^p_{\mathrm{NR}}(y,x)$ for all $x, y \in \mathbb{R}^n$. Furthermore, $\varphi^p_{\mathrm{NR}}(x,y)$ is continuously differentiable.

Proposition 3.2. *Let φ^p_{NR} be defined as in (7) with $p > 1$ being a positive odd integer and $g^{\mathrm{soc}}(z) = z^p$, $h^{\mathrm{soc}}(z) = |z|^p$ be the vector-valued functions corresponding to $g(t) = t^p$ and $h(t) = |t|^p$ for $t \in \mathbb{R}$, respectively. Then, φ^p_{NR} is continuously differentiable at any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover, we have*

$$\begin{aligned} \nabla_x \varphi^p_{\mathrm{NR}}(x,y) &= \frac{1}{2^p}\left(\nabla g^{\mathrm{soc}}(w) - \nabla h^{\mathrm{soc}}(v)\right), \\ \nabla_y \varphi^p_{\mathrm{NR}}(x,y) &= \frac{1}{2^p}\left(\nabla g^{\mathrm{soc}}(w) + \nabla h^{\mathrm{soc}}(v)\right), \end{aligned}$$

where $v := v(x,y) = x + y$, $w := w(x,y) = x - y$, and

$$\nabla g^{\mathrm{soc}}(v) = \begin{cases} p v_1^{p-1} I & \text{if} \quad v_2 = 0; \\ \begin{bmatrix} b_1(v) & c_1(v)\bar{v}_2^T \\ c_1(v)\bar{v}_2 & a_1(v)I + (b_1(v) - a_1(v))\,\bar{v}_2\bar{v}_2^T \end{bmatrix} & \text{if} \quad v_2 \neq 0; \end{cases}$$

$$\begin{aligned} \bar{v}_2 &= \frac{v_2}{\|v_2\|}, \\ a_1(v) &= \frac{(\lambda_2(v))^p - (\lambda_1(v))^p}{\lambda_2(v) - \lambda_1(v)}, \\ b_1(v) &= \frac{p}{2}\left[(\lambda_2(v))^{p-1} + (\lambda_1(v))^{p-1}\right], \\ c_1(v) &= \frac{p}{2}\left[(\lambda_2(v))^{p-1} - (\lambda_1(v))^{p-1}\right], \end{aligned}$$

and

$$\nabla h^{\rm soc}(w) = \begin{cases} p w_1 |w_1|^{p-2} I & \text{if} \quad w_2 = 0; \\ \begin{bmatrix} b_2(w) & c_2(w)\bar{w}_2^T \\ c_2(w)\bar{w}_2 & a_2(w)I + (b_2(w) - a_2(w))\,\bar{w}_2\bar{w}_2^T \end{bmatrix} & \text{if} \quad w_2 \neq 0; \end{cases}$$

$$\begin{aligned} \bar{w}_2 &= \frac{w_2}{\|w_2\|}, \\ a_2(w) &= \frac{|\lambda_2(w)|^p - |\lambda_1(w)|^p}{\lambda_2(w) - \lambda_1(w)}, \\ b_2(w) &= \frac{p}{2}\left[\lambda_2(w)|\lambda_2(w)|^{p-2} + \lambda_1(w)|\lambda_1(w)|^{p-2}\right], \\ c_2(w) &= \frac{p}{2}\left[\lambda_2(w)|\lambda_2(w)|^{p-2} - \lambda_1(w)|\lambda_1(w)|^{p-2}\right], \end{aligned}$$

Proof. From the definition of $\varphi^p_{\rm NR}$, it is clear to see that for any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} \varphi^p_{\rm NR}(x,y) &= \frac{1}{2^p}\left[(\lambda_1(v))^p u_v^{(1)} + (\lambda_2(v))^p u_v^{(2)}\right] - \frac{1}{2^p}\left[|\lambda_1(w)|^p u_w^{(1)} + |\lambda_2(w)|^p u_w^{(2)}\right] \\ &= \frac{1}{2^p}\left(g^{\rm soc}(v) - h^{\rm soc}(w)\right). \end{aligned} \tag{8}$$

For $p \geq 3$, since both $|t|^p$ and t^p are continuously differentiable on $\mathbb{R}$, by [5, Proposition 5] and [19, Proposition 5.2], we know that the function $g^{\rm soc}$ and $h^{\rm soc}$ are continuously differentiable on $\mathbb{R}^n$. Moreover, it is clear that $v(x,y) = x + y$, $w(x,y) = x - y$ are continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n$, then we conclude that $\varphi^p_{\rm NR}$ is continuously differentiable. Moreover, from the formula in [5, Proposition 4] and [19, Proposition 5.2], we have

$$\nabla g^{\rm soc}(v) = \begin{cases} p v_1^{p-1} I & \text{if} \quad v_2 = 0; \\ \begin{bmatrix} b_1(v) & c_1(v)\bar{v}_2^T \\ c_1(v)\bar{v}_2 & a_1(v)I + (b_1(v) - a_1(v))\,\bar{v}_2\bar{v}_2^T \end{bmatrix} & \text{if} \quad v_2 \neq 0; \end{cases}$$

$$\nabla h^{\rm soc}(w) = \begin{cases} p w_1 |w_1|^{p-2} I & \text{if} \quad w_2 = 0; \\ \begin{bmatrix} b_2(w) & c_2(w)\bar{w}_2^T \\ c_2(w)\bar{w}_2 & a_2(w)I + (b_2(w) - a_2(w))\,\bar{w}_2\bar{w}_2^T \end{bmatrix} & \text{if} \quad w_2 \neq 0; \end{cases}$$

where

$$\begin{array}{ll} \bar{v}_2 = \frac{v_2}{\|v_2\|}, & \bar{w}_2 = \frac{w_2}{\|w_2\|} \\ a_1(v) = \frac{(\lambda_2(v))^p - (\lambda_1(v))^p}{\lambda_2(v) - \lambda_1(v)}, & a_2(w) = \frac{|\lambda_2(w)|^p - |\lambda_1(w)|^p}{\lambda_2(w) - \lambda_1(w)}, \\ b_1(v) = \frac{p}{2}\left[(\lambda_2(v))^{p-1} + (\lambda_1(v))^{p-1}\right], & b_2(w) = \frac{p}{2}\left[\lambda_2(w)|\lambda_2(w)|^{p-2} + \lambda_1(w)|\lambda_1(w)|^{p-2}\right], \\ c_1(v) = \frac{p}{2}\left[(\lambda_2(v))^{p-1} - (\lambda_1(v))^{p-1}\right], & c_2(w) = \frac{p}{2}\left[\lambda_2(w)|\lambda_2(w)|^{p-2} - \lambda_1(w)|\lambda_1(w)|^{p-2}\right]. \end{array}$$

By taking differentiation on both sides about x and y for (8), respectively, and applying the chain rule for differentiation, it follows that

$$\begin{aligned} \nabla_x \varphi^p_{\rm NR}(x,y) &= \frac{1}{2^p}\left(\nabla g^{\rm soc}(w) - \nabla h^{\rm soc}(v)\right), \\ \nabla_y \varphi^p_{\rm NR}(x,y) &= \frac{1}{2^p}\left(\nabla g^{\rm soc}(w) + \nabla h^{\rm soc}(v)\right), \end{aligned}$$

Hence, we complete the proof. $\square$

Lastly, we collect a list of known complementarity functions associated with second-order cone. Especially, some of them could be employed to symmetric cone complementary problems. Applying Proposition 3.1 may create more new complementarity functions.

1. $\phi_{\mathrm{NR}}(x,y) = x - (x-y)_+$ (Fukushima, Luo, and Tseng [19]),
2. $\phi_{\mathrm{FB}}(x,y) = (x^2+y^2)^{1/2} - x - y$ (Fukushima, Luo, and Tseng [19]),
3. $\phi_{\mathrm{CSS}}(x,y) = x - (x-y)_+ + (x)_+ \circ (y)_+$ (Chen, Sun, and Sun [12]),
4. $\phi_{\mathrm{MS}}(x,y) = x \circ y + \frac{1}{2\alpha}\left\{[(x-\alpha y)_+]^2 - x^2 + [(y-\alpha x)_+]^2 - y^2\right\}$, $\alpha > 1$ (Kong, Tunçel, and Xiu [28]),
5. $\phi_{\mathrm{KK}}(x,y) = \left[(x-y)^2 + \tau(x \circ y)\right]^{1/2} - x - y$, $\tau \in (0,4)$ (Chen and Pan [10]),
6. $\phi_{\mathrm{TLM}}(x,y) = (x)_+ \circ (y)_+ + [(x)_-]^2 + [(y)_-]^2$ (Tang, Liu, and Ma [35]),
7. $\phi_{\mathrm{PNR}}(x,y) = \lambda\phi_{\mathrm{NR}}(x,y) + (1-\lambda)[(x)_+ \circ (y)_+]$, $\lambda \in (0,1)$ (Kum, and Lim [29]),
8. $\phi_{\mathrm{PFB}}(x,y) = \lambda\phi_{\mathrm{FB}}(x,y) + (1-\lambda)[(x)_+ \circ (y)_+]$, $\lambda \in (0,1)$ (Kum, and Lim [29]),
9. $\phi_{\mathrm{PGFB}}(x,y) = \left[\lambda(|x|^p + |y|^p) + (1-\lambda)|x+y|^p\right]^{1/p} - x - y$, $\lambda \in (0,2), p = 2$ or $\lambda \in (0,1], p \in (1,2) \cup (2,+\infty)$ (Hu, Huang, and Lu [23]),
10. $\phi_{\mathrm{EP1}}(x,y) = -(x \circ y) + \frac{1}{2\alpha}[(x+y)_+]^2$, $\alpha \in (0,1]$ (Chen and Pan [11]),
11. $\phi_{\mathrm{EP2}}(x,y) = -(x \circ y) + \frac{1}{2\beta}\{[(x)_-]^2 + [(y)_-]^2\}$, $\beta \in (0,1)$ (Chen and Pan [11]),
12. $\phi^p_{\mathrm{FB}}(x,y) = (|x|^p + |y|^p)^{1/p} - x - y$, $p \in (1,+\infty)$ (Pan, Kum, Lim, and Chen [33]),
13. $\phi^p_{\mathrm{NR}}(x,y) = x^p - [(x-y)_+]^p$, $p > 1$ is odd integer (Ma, Chen, Huang, and Ko [30]),
14. $\phi^p_{\mathrm{D-FB}}(x,y) = (x^2+y^2)^{p/2} - (x+y)^p$, $p > 1$ is odd integer (Ma, Chen Huang, and Ko [30]).

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